

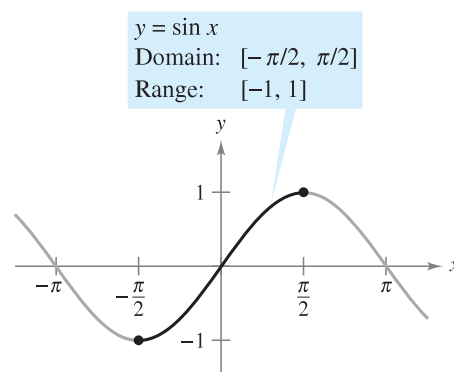
Section 5.6 Inverse Trigonometric Functions: Differentiation**Inverse Trigonometric Functions**

This section begins with a rather surprising statement: *None of the six basic trigonometric functions has an inverse function.* This statement is true because all six trigonometric functions are periodic and therefore are not one-to-one. In this section you will examine these six functions to see whether their domains can be redefined in such a way that they will have inverse functions on the *restricted domains*.

In Example 4 of Section 5.3, you saw that the sine function is increasing (and therefore is one-to-one) on the interval  $[-\pi/2, \pi/2]$  (see Figure 5.28). On this interval you can define the inverse of the *restricted* sine function as

$$y = \arcsin x \quad \text{if and only if} \quad \sin y = x$$

where  $-1 \leq x \leq 1$  and  $-\pi/2 \leq \arcsin x \leq \pi/2$ .



The sine function is one-to-one on  $[-\pi/2, \pi/2]$ .

**Figure 5.28**

**NOTE** The term “iff” is used to represent the phrase “if and only if.”

Under suitable restrictions, each of the six trigonometric functions is one-to-one and so has an inverse function, as shown in the following definition.

#### Definitions of Inverse Trigonometric Functions

Function	Domain	Range
$y = \arcsin x$ iff $\sin y = x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \arccos x$ iff $\cos y = x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$y = \arctan x$ iff $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \text{arccot } x$ iff $\cot y = x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = \text{arcsec } x$ iff $\sec y = x$	$ x  \geq 1$	$0 \leq y \leq \pi, y \neq \frac{\pi}{2}$
$y = \text{arccsc } x$ iff $\csc y = x$	$ x  \geq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$

**NOTE** The term “arcsin  $x$ ” is read as “the arcsine of  $x$ ” or sometimes “the angle whose sine is  $x$ .” An alternative notation for the inverse sine function is “ $\sin^{-1} x$ .” ■

The graphs of the six inverse trigonometric functions are shown in Figure 5.29.

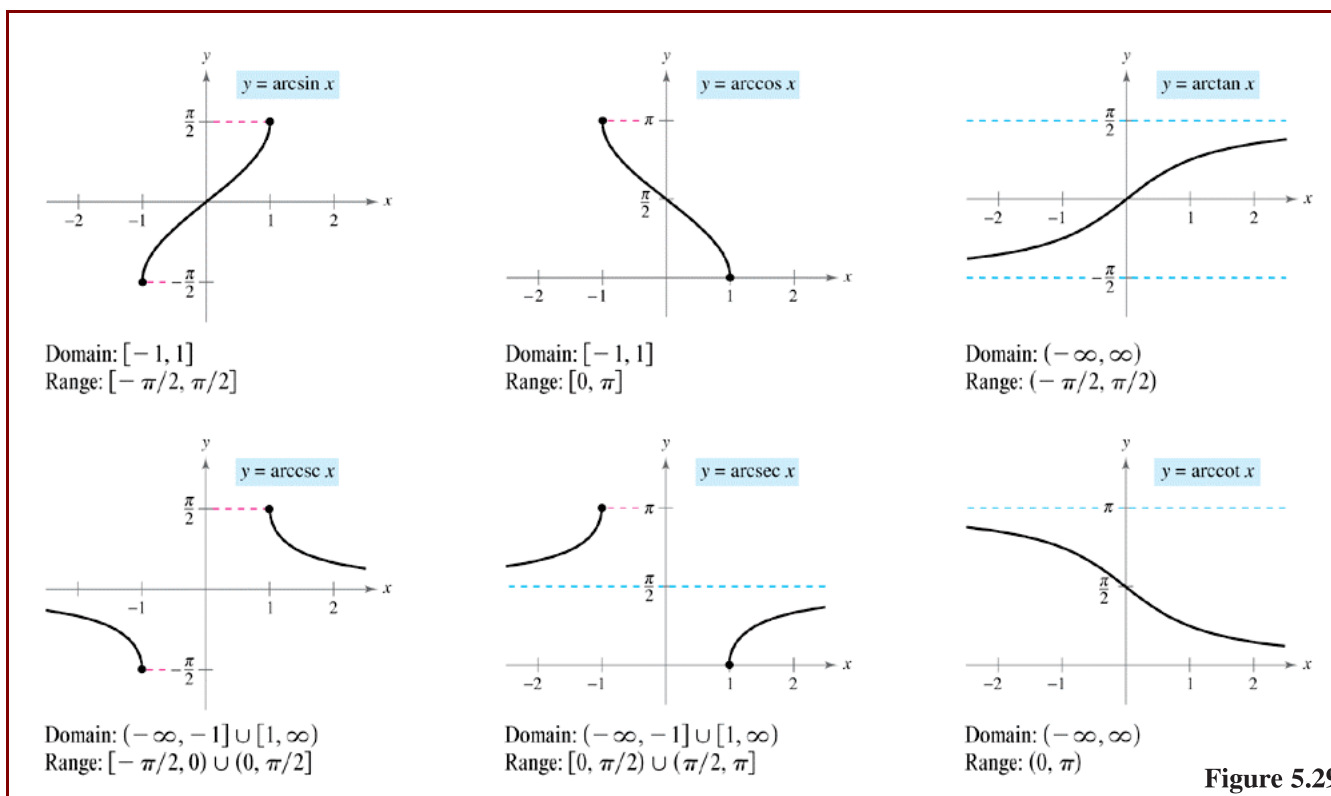


Figure 5.29

### Ex.1 Evaluating Inverse Trigonometric Functions

Evaluate each function.

- a.  $\arcsin\left(-\frac{1}{2}\right)$     b.  $\arccos 0$     c.  $\arctan \sqrt{3}$     d.  $\arcsin(0.3)$

**NOTE** When evaluating inverse trigonometric functions, remember that they denote *angles in radian measure*.

Inverse functions have the properties

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

When applying these properties to inverse trigonometric functions, remember that the trigonometric functions have inverse functions only in restricted domains. For  $x$ -values outside these domains, these two properties do not hold. For example,  $\arcsin(\sin \pi)$  is equal to 0, not  $\pi$ .

#### Properties of Inverse Trigonometric Functions

If  $-1 \leq x \leq 1$  and  $-\pi/2 \leq y \leq \pi/2$ , then

$$\sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin y) = y.$$

If  $-\pi/2 < y < \pi/2$ , then

$$\tan(\arctan x) = x \quad \text{and} \quad \arctan(\tan y) = y.$$

If  $|x| \geq 1$  and  $0 \leq y < \pi/2$  or  $\pi/2 < y \leq \pi$ , then

$$\sec(\operatorname{arcsec} x) = x \quad \text{and} \quad \operatorname{arcsec}(\sec y) = y.$$

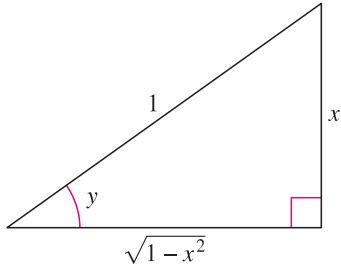
Similar properties hold for the other inverse trigonometric functions.

#### Ex.2 Solving an Equation

Solve  $\arctan(2x - 3) = \frac{\pi}{4}$

### Ex.3 Using Right Triangles

- Given  $y = \arcsin x$ , where  $0 < y < \pi/2$ , find  $\cos y$ .
- Given  $y = \operatorname{arcsec}(\sqrt{5}/2)$ , find  $\tan y$ .



$y = \arcsin x$   
Figure 5.30

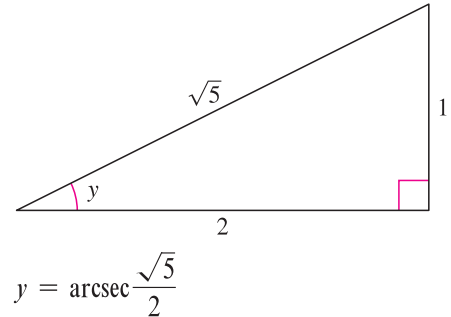


Figure 5.31

## Derivatives of Inverse Trigonometric Functions

In Section 5.1 you saw that the derivative of the *transcendental* function  $f(x) = \ln x$  is the *algebraic* function  $f'(x) = 1/x$ . You will now see that the derivatives of the inverse trigonometric functions also are algebraic (even though the inverse trigonometric functions are themselves transcendental).

The following theorem lists the derivatives of the six inverse trigonometric functions. Proofs for  $\arcsin u$  and  $\arccos u$  are given in Appendix A, and the rest are left as an exercise. (See Exercise 104.) Note that the derivatives of  $\arccos u$ ,  $\operatorname{arccot} u$ , and  $\operatorname{arccsc} u$  are the *negatives* of the derivatives of  $\arcsin u$ ,  $\operatorname{arctan} u$ , and  $\operatorname{arcsec} u$ , respectively.

**THEOREM 5.16 Derivatives of Inverse Trigonometric Functions**

Let  $u$  be a differentiable function of  $x$ .

$$\begin{aligned} \frac{d}{dx} [\arcsin u] &= \frac{u'}{\sqrt{1-u^2}} & \frac{d}{dx} [\arccos u] &= \frac{-u'}{\sqrt{1-u^2}} \\ \frac{d}{dx} [\arctan u] &= \frac{u'}{1+u^2} & \frac{d}{dx} [\operatorname{arccot} u] &= \frac{-u'}{1+u^2} \\ \frac{d}{dx} [\operatorname{arcsec} u] &= \frac{u'}{|u|\sqrt{u^2-1}} & \frac{d}{dx} [\operatorname{arccsc} u] &= \frac{-u'}{|u|\sqrt{u^2-1}} \end{aligned}$$

**NOTE** There is no common agreement on the definition of  $\operatorname{arcsec} x$  (or  $\operatorname{arccsc} x$ ) for negative values of  $x$ . When we defined the range of the arcsecant, we chose to preserve the reciprocal identity

$$\operatorname{arcsec} x = \arccos \frac{1}{x}.$$

For example, to evaluate  $\operatorname{arcsec}(-2)$ , you can write

$$\operatorname{arcsec}(-2) = \arccos(-0.5) \approx 2.09.$$

One of the consequences of the definition of the inverse secant function given in this text is that its graph has a positive slope at every  $x$ -value in its domain. (See Figure 5.29.) This accounts for the absolute value sign in the formula for the derivative of  $\operatorname{arcsec} x$ .

**TECHNOLOGY** If your graphing utility does not have the arcsecant function, you can obtain its graph using

$$f(x) = \operatorname{arcsec} x = \arccos \frac{1}{x}.$$

**Ex.4 Differentiating Inverse Trigonometric Functions**

a.  $\frac{d}{dx} [\arcsin(2x)] =$

b.  $\frac{d}{dx} [\arctan(3x)] =$

c.  $\frac{d}{dx} [\arcsin \sqrt{x}] =$  .

d.  $\frac{d}{dx} [\operatorname{arcsec} e^{2x}] =$

**Ex.5** A Derivative That Can Be Simplified

$$y = \arcsin x + x\sqrt{1 - x^2}$$

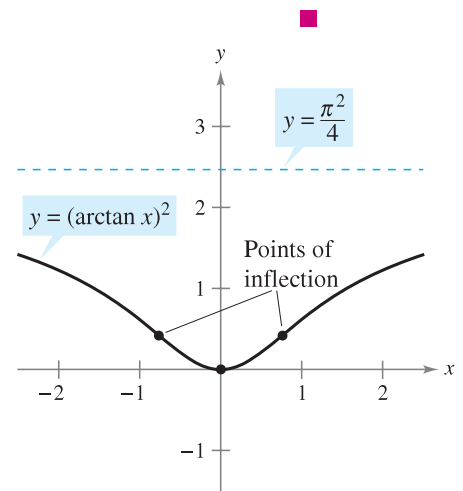
Find  $y' =$  .

**NOTE** From Example 5, you can see one of the benefits of inverse trigonometric functions—they can be used to integrate common algebraic functions. For instance, from the result shown in the example, it follows that

$$\int \sqrt{1-x^2} dx = \frac{1}{2}(\arcsin x + x\sqrt{1-x^2}).$$

### Ex.6 Analyzing an Inverse Trigonometric Graph

Analyze the graph of  $y = (\arctan x)^2$ .



The graph of  $y = (\arctan x)^2$  has a horizontal asymptote at  $y = \pi^2/4$ .

**Figure 5.32**

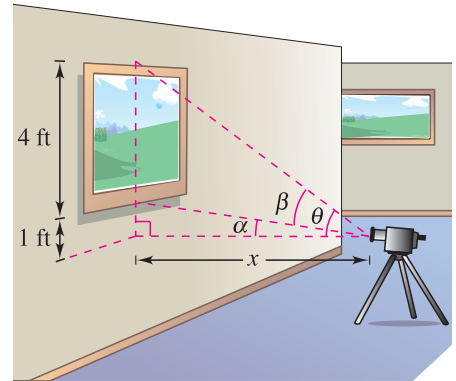


### Ex.7 Maximizing an Angle

A photographer is taking a picture of a painting hung in an art gallery. The height of the painting is 4 feet. The camera lens is 1 foot below the lower edge of the painting, as shown in Figure 5.33. How far should the camera be from the painting to maximize the angle subtended by the camera lens?

**Solution** In Figure 5.33, let  $\beta$  be the angle to be maximized.

$$\begin{aligned}\beta &= \theta - \alpha \\ &= \operatorname{arccot} \frac{x}{5} - \operatorname{arccot} x\end{aligned}$$



The camera should be 2.236 feet from the painting to maximize the angle  $\beta$ .

**Figure 5.33**



## Review of Basic Differentiation Rules

As mathematics has developed during the past few hundred years, a small number of elementary functions have proven sufficient for modeling most\* phenomena in physics, chemistry, biology, engineering, economics, and a variety of other fields. An **elementary function** is a function from the following list or one that can be formed as the sum, product, quotient, or composition of functions in the list.

### Algebraic Functions

Polynomial functions

Rational functions

Functions involving radicals

### Transcendental Functions

Logarithmic functions

Exponential functions

Trigonometric functions

Inverse trigonometric functions

With the differentiation rules introduced so far in the text, you can differentiate *any* elementary function. For convenience, these differentiation rules are summarized below.

### Basic Differentiation Rules for Elementary Functions

$$1. \frac{d}{dx}[cu] = cu'$$

$$2. \frac{d}{dx}[u \pm v] = u' \pm v'$$

$$3. \frac{d}{dx}[uv] = uv' + vu'$$

$$4. \frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$$

$$5. \frac{d}{dx}[c] = 0$$

$$6. \frac{d}{dx}[u^n] = nu^{n-1}u'$$

$$7. \frac{d}{dx}[x] = 1$$

$$8. \frac{d}{dx}[|u|] = \frac{u}{|u|}(u'), \quad u \neq 0$$

$$9. \frac{d}{dx}[\ln u] = \frac{u'}{u}$$

$$10. \frac{d}{dx}[e^u] = e^u u'$$

$$11. \frac{d}{dx}[\log_a u] = \frac{u'}{(\ln a)u}$$

$$12. \frac{d}{dx}[a^u] = (\ln a)a^u u'$$

$$13. \frac{d}{dx}[\sin u] = (\cos u)u'$$

$$14. \frac{d}{dx}[\cos u] = -(\sin u)u'$$

$$15. \frac{d}{dx}[\tan u] = (\sec^2 u)u'$$

$$16. \frac{d}{dx}[\cot u] = -(\csc^2 u)u'$$

$$17. \frac{d}{dx}[\sec u] = (\sec u \tan u)u'$$

$$18. \frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$$

$$19. \frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$$

$$20. \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$21. \frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$$

$$22. \frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$$

$$23. \frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}}$$

$$24. \frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$